Journal of Statistical Physics, Vol. 86, Nos. 5/6, 1997

Effect of the Block-Spin Configuration on the Location of β_c in Two-Dimensional Ising Models

Mohamed Ould-Lemrabott[1]

Received April 2, 1996; final June 28, 1996

We consider the nearest neighbor Ising model on the 2D square lattice and divide the lattice into 2 by 2 blocks. Each block is assigned one spin value (1 or -1) and these block spin values are kept fixed. We then impose the majority rule and look at the effect on the phase transition that was present in the original unconstrained spin system. We find that for the checkerboard block-spin configuration, Monte Carlo simulations show that β_c is close to 1, which, compared to the original nearest neighbor Ising $\beta_c = 0.44...$, shows that the critical temperature has been reduced by more than one half. For none of the other 11 block-spin configurations that we have considered is there any indication of a phase transition in the constrained system of original spins.

KEY WORDS: Lattice spin system; majority rule; Monte Carlo simulation; renormalization group transformations.

1. INTRODUCTION

In the renormalization group theory, there is a fundamental assumption that the introduction of the block spins into a critical system should make the correlation length finite. In other words, if a choice of the block-spin configuration is made and these block spins are fixed, then the constrained system should not have a phase transition at the same temperature as the original system. Rather, the critical point should be shifted to a lower temperature, i.e., larger β (inverse temperature). Kennedy proved that in 2D for three choices of the block-spin configuration, one of them being the checkerboard, the majority rule imposed on the original spins does cause the critical temperature to shift to the left.[5] Benfatto *et al.*[1] ran Monte Carlo simulations of a renormalization group transformation in which the

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[1] Program in Applied Mathematics, University of Arizona, Tucson, Arizona 85721; e-mail: ould@math.arizona.edu.

0022-4715/97/0300-1109$12.50/0

block-spin is taken to be the sum of spins in the block. They considered the block-spin configuration in which all the block spins are zero. They found that the introduction of this block spin configuration does indeed lower the critical temperature, but only by about 10%. We will show by numerical computation that the majority rule gives rise to a substantially larger shift in the critical temperature than found in ref. 1.

Van Enter *et al.*[3] have proven that, at low enough temperature, there is a phase transition for 7 by 7 blocks with the doubly alternating block-spin configuration pictured in Fig. 4. In their argument, they proved that the constrained system has two Gibbs states. We find that for 2 by 2 blocks, the number of ground states grows exponentially with the lattice size. For a recent review of these pathologies that can occur at low temperature, see ref. 4 and references therein. Cirillo and Olivieri[2] recently studied similar constrained models with a slightly different type of majority rule, where for each of the two values of a block spin (1 and -1), there are eight compatible blocks, whereas in our case there are 11 possible blocks compatible with each block spin value.

We consider the nearest neighbor Ising model on a lattice, which is defined by

$$H(\sigma) = -\sum_{\langle ij \rangle} \sigma_i \sigma_j$$

where $\langle ij \rangle$ denotes nearest neighbor sites. Each spin σ_i takes on one of two values in $\{-1, 1\}$. We define the following probability measure on the space of spin configurations

$$\pi(\sigma) = \frac{e^{-\beta H(\sigma)}}{Z}$$

where Z is the constant that makes π a probability measure, and β is the inverse temperature. This is the probability measure for the unconstrained Ising model. We will simulate this measure to obtain the well-known β_c for this model. We divide the lattice into 2 by 2 blocks and assign a spin value to each block, which we call the block-spin value. The spins in the original system are denoted by σ. The "block" spins are denoted by σ'. Each block can take on one of 16 configurations: 6 are tied blocks, 5 have a majority of $+$, and 5 have a majority of $-$. We consider real space renormalization group transformations for such systems, which are formally defined by

$$e^{-H'(\sigma')} = \sum_{\sigma} T(\sigma, \sigma')\, e^{-\beta H(\sigma)}$$

where $T(\sigma, \sigma')$ is a probability kernel, i.e., $\sum_{\sigma'} T(\sigma, \sigma') = 1$, for every configuration σ. For each choice of the block-spin configuration σ', let $\pi_{\sigma'}$ be the measure on spin configurations σ which, for the finite-size lattices, is given by taking the probability of σ to be proportional to $T(\sigma, \sigma')^{e - H(\sigma)}$. We take the kernel $T(\sigma, \sigma')$ to be that of the majority rule transformation. Introductory discussions of these renormalization group transformations may be found in refs. 3 and 6. Numerical results show that there is a shift in the β_c for a particular choice of a configuration of the block spins, namely the checkerboard configuration.

2. NUMERICAL COMPUTATIONS

For each block-spin configuration σ' we use Monte Carlo methods to simulate a Markov chain whose equilibrium distribution is

$$\pi_{\sigma'}(\sigma) = \frac{e^{-E(\sigma)}}{Z_{\sigma'}}$$

where

$$E(\sigma) = \beta H(\sigma) + \ln(2)\, N(\sigma)$$

where $N(\sigma)$ denotes the number of tied block spins in the original spin configuration σ and

$$Z_{\sigma'} = \sum_{\sigma:\sigma'} e^{-E(\sigma)}$$

where $\sigma:\sigma'$ means that each block in σ is either tied or a majority of its spins agree with the corresponding block-spin in σ'; we will refer to this situation by "σ is compatible with σ'." In situations where σ is not compatible with σ', the probability measure $\pi_{\sigma'}$ takes the value 0. At each block site in the "renormalized" lattice, which we denote by Λ', we compute the change in energy ΔE that would be caused by "flipping" the block, i.e., assigning to it one of the 11 block configurations that agree with the block spin. If ΔE is negative, we accept the proposed "flip." If ΔE is positive, we accept the proposed configuration with probability $e^{-\Delta E}$. After a "sweep" of the lattice, we update the chosen observable, in this case the specific heat:

$$C_{t} = \beta^2 |\Lambda|^{-1} (\langle H^2 \rangle - \langle H \rangle^2)$$

where $|\Lambda|$ is the number of block spins in the original lattice. Throughout each Monte Carlo simulation, the block-spin configuration is unchanged, only the blocks themselves are subject to change. Since these simulations are taking place on a finite lattice, we define the boundary terms periodically.

We start the simulation with a random configuration of the blocks which is in agreement with the block spin configuration, and we run 40,000 lattice sweeps before we start updating the observable, to account for the initial bias occurring from starting probably far from equilibrium. All the results shown here were obtained with 400,000 sweeps of the lattice. To get an estimate of the specific heat for a particular volume, we compute its average over 20,000 sweeps of the lattice Λ', then we compute the average of the resulting 20 values. This procedure allows us to estimate the standard deviation of the estimates. As an example, for the checkerboard constrained system, for $L=32$, the estimated variance of the estimated specific heat starts at 0.116924 for $\beta=0.92$, then increases to 5.729983 for $\beta=1.0$, which is near the point of criticality and then decreases to 0.076778 for $\beta=1.06$.

We considered 12 periodic block spin configurations that can be thought of as natural. The patterns used for these configurations are pictured in Fig. 4. The configuration for the whole finite lattice is obtained by repeating the patterns as many times as necessary. The volumes are always taken to be of the type $4k$ by $4l$ and in all our applications $k=l$.

First, note that the lengths indicated on the figures are meant in terms of block spins, i.e., an 8 by 8 lattice contains 64 block spins and 256 original spins. The simulation pictured in Fig. 1 represents the graph of the estimated specific heat for the regular unconstrained nearest neighbor Ising model. The peak in the specific heat, which is located near $\beta_c=0.44\ldots$, shows a strong volume dependence. This leads one to conclude that the infinite volume limit of the specific heat will have a discontinuity at β_c. A similar observation can be made regarding the checkerboard configuration (Fig. 2), where the critical β is near 1. However, in Fig. 3, which represents the specific heat for the doubly alternating configuration considered in ref. 3, something very different is happening in the sense that here the specific heat curve is almost independent of the volume size. Thus, we conclude the absence of a phase transition for this particular block-spin configuration. A similar behavior was found for some of the other configurations, for which the result of the simulations is not shown. Other configurations show a peak which is clearly not of the same nature as for the checkerboard configuration.

The striking fact about the shift in β_c that we have encountered is that the smallest one, found for the checkerboard configuration, is much bigger than the type of shift found in ref. 1. The nearest neighbor Ising model without any constraints has two ground states which give rise to two Gibbs

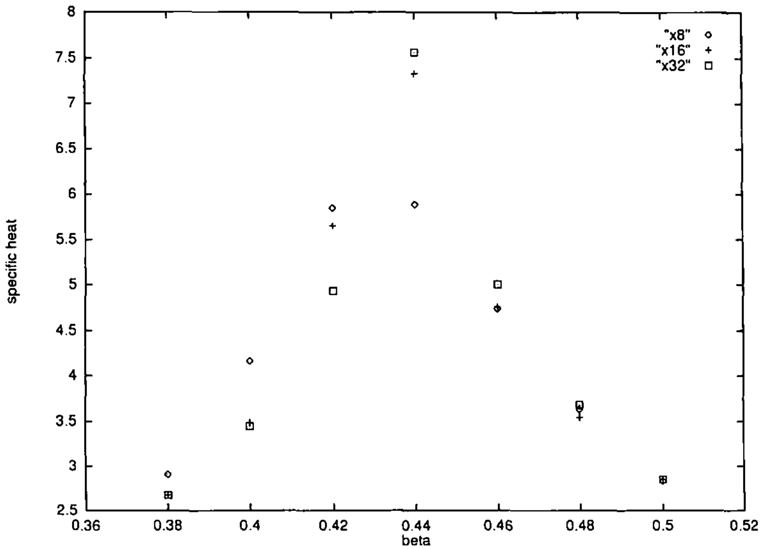

Fig. 1. Unconstrained nn Ising model, $L=8, 16, 32$, 440 K sweeps.

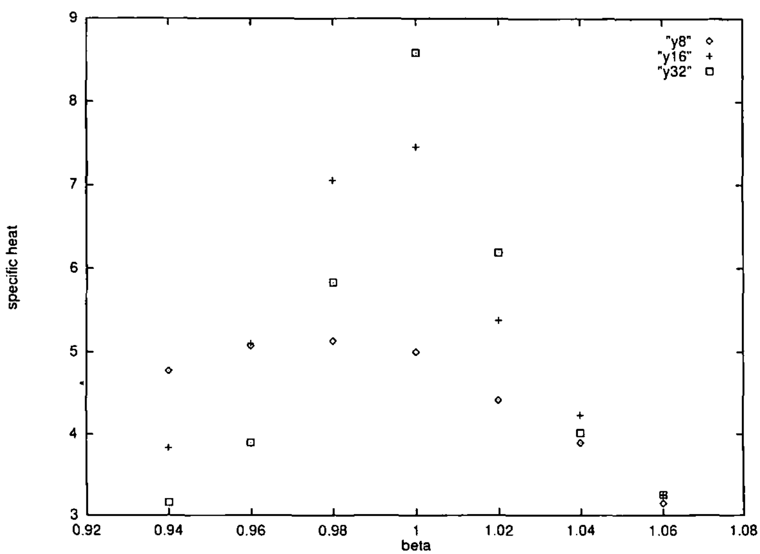

Fig. 2. Checkerboard configuration, $L=8, 16, 32$, 440 K sweeps.

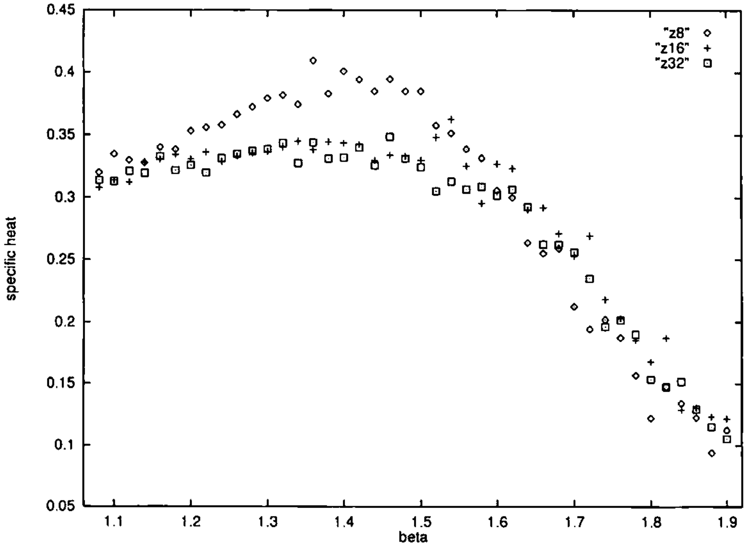

Fig. 3. Doubly alternating configuration from ref. 3, $L = 8, 16, 32$, 440 K sweeps.

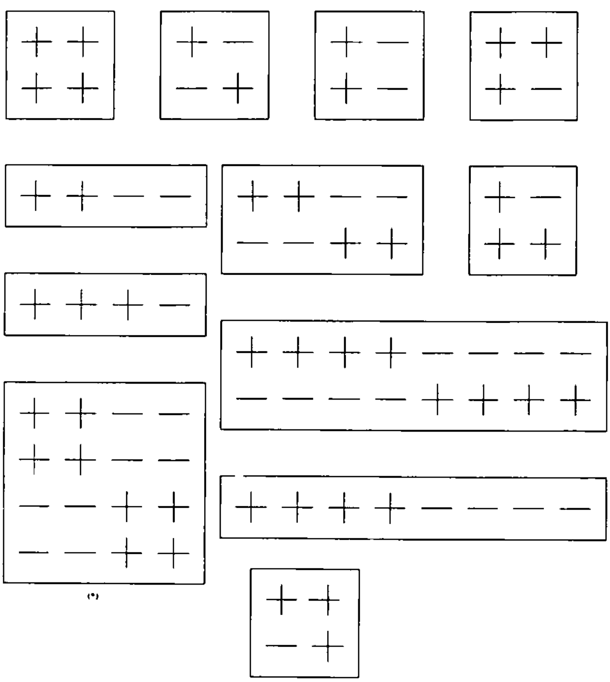

Fig. 4. The 12-spin block configurations. Here (*) indicates the configuration used in ref. 3.

states at low temperature, and hence a phase transition. For the checkerboard-configuration constrained model, there are four ground states that can be obtained from each other by a natural symmetry. A phase transition was found for this model. However, for the doubly alternating configuration, the number of ground states grows exponentially with the lattice size. In this case, no phase transition was found. None of the periodic constraints we have considered has phase transition at a temperature which is higher than half the critical temperature of the unconstrained model. This supports the suggestion of ref. 5 that the majority rule transformation is well defined in a region including the critical temperature.

ACKNOWLEDGMENTS

I am very grateful to my Ph.D. thesis advisor, Prof. Tom Kennedy, for his kind and generous help, without which this work would never have been completed. I also would like to express my gratitude to the U.S. Agency for International Development and the African American Institute for sponsoring most of my studies in this country. This work was supported in part by NSF grant DMS-9303051.

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